

ON SERIAL SYMMETRIC EXCHANGES OF MATROID BASES

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ABSTRACT. We study some properties of a serial (i.e. one-by-one) symmetric exchange of elements of two disjoint bases of a matroid. We show that any two elements of one base have a serial symmetric exchange with some two elements of the other base. As a result, we obtain that any two disjoint bases in a matroid of rank 4 have a full serial symmetric exchange.

1. INTRODUCTION

A matroid is a hereditary family M of subsets (called independent) of a finite ground set S that satisfies an exchange axiom: If $A, B \in M$ and $|B| > |A|$, then there exists $x \in B \setminus A$ such that $A \cup x \in M$. A maximal independent set is called a *base*. An element $x \in S$ is spanned by A if either $x \in A$ or $I \cup \{x\} \notin M$ for some independent set $I \subseteq A$. The rank of $A \subseteq S$, denoted here by $\rho(A)$, is the size of a maximal independent subset in A . We also adopt the common notation $A + x$ for $A \cup \{x\}$ and $A - x$ for $A \setminus \{x\}$. A *circuit* is a minimal dependent set. When I is independent but $I + x$ is not, we shall denote the unique minimal subset of I that spans x (called the *support* of x) by $C(I, x)$. We denote by $C^+(I, x)$ the circuit $C(I, x) + x$. For further knowledge and details about matroid theory the reader is referred to Oxley [10] and Welsh [13].

The main goal of this paper is to examine the following conjecture:

Conjecture 1.1. *Let B_1 and B_2 be two disjoint bases of a matroid M of rank n . There exists an ordering $\{b_1 \prec b_2 \prec \dots \prec b_{2n}\}$ of the elements of $B_1 \cup B_2$, such that the first n elements belong to B_1 and, for every $i = 1, 2, \dots, 2n$, the set $\{b_{i \bmod 2n}, b_{(i+1) \bmod 2n}, \dots, b_{(i+n-1) \bmod 2n}\}$ is a base.*

As far as we know, this intriguing conjecture was posed implicitly in an early paper by Gabow [6], and later in the more general form of “cyclic base orders” by Kajitani and Sugishta [8] and Weidemann [12]. Partial results and some further possible generalizations appear in the works of Kajitani et al. [9], Cordovil and Moreira [3] and van den Heuvel and Thomassé [11]. Recently Bonin [1] proved it for sparse paving matroids.

It is easy to see that Conjecture 1.1 may be reformulated in the following equivalent form:

Conjecture 1.2. *Let A and B be two disjoint bases of a matroid M of rank n . There exists an ordering $\{a_1 \prec a_2 \prec \dots \prec a_n\}$ of the elements of A and an ordering*

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$\{b_1 \prec b_2 \prec \dots \prec b_n\}$ of the elements of B , such that for every $i = 1, 2, \dots, n$ both $(A \setminus \{a_1, \dots, a_i\}) \cup \{b_1, \dots, b_i\}$ and $(B \setminus \{b_1, \dots, b_i\}) \cup \{a_1, \dots, a_i\}$ are bases of M .

In [3] the same problem was cast in terms of the “base-cobase graph”:

Definition 1.3. A matroid M whose ground set S is a disjoint union of two bases is called a *block matroid*. The *base-cobase graph* $G(V, E)$ of a block matroid M consists of a set of vertices $V = \{B \in M \mid B \text{ and } S \setminus B \text{ are bases}\}$, where the unordered pair (B, B') is an edge if and only if B and B' are bases in V differing by exactly two elements, i.e. $|B \triangle B'| = 2$.

Under these terms, Conjecture 1.2, restricted to block matroids, takes the following form [3]:

Conjecture 1.4. *If G is the base-cobase graph of a block matroid of rank n , then the diameter of G is equal to n .*

Conjecture 1.4 was proved for graphic block matroids by Farber, Richter and Shank [5] (with a modification by Weidemann [12]), and independently by Kajitani et al. [9] and Cordovil and Moreira [3]. It was also proved for transversal block matroids by Farber [4]. As far as we are aware, it is still unknown whether the base-cobase graph is connected for all block matroids.

2. SOME DEFINITIONS AND LEMMAS

We begin with a few definitions and notations, some of which were introduced by Gabow [6]. For convenience, we slightly modify the terminology used there. Let M be a matroid of rank n , and let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ be two disjoint bases in M .

Definition 2.1. Let $X \subseteq A$ and $Y \subseteq B$ such that $|X| = |Y| = k$.

(i) The pair (X, Y) is a *serial exchange relative to the base A* if there exist orderings $X = \{a_1 \prec a_2 \prec \dots \prec a_k\}$ and $Y = \{b_1 \prec b_2 \prec \dots \prec b_k\}$ so that for each $i = 1, 2, \dots, k$, $(A \setminus \{a_1, \dots, a_i\}) \cup \{b_1, \dots, b_i\}$ is a base.

(ii) The pair (X, Y) is a *serial symmetric exchange relative to the bases A and B* if there exist orderings $X = \{a_1 \prec a_2 \prec \dots \prec a_k\}$ and $Y = \{b_1 \prec b_2 \prec \dots \prec b_k\}$ so that for each $i = 1, 2, \dots, k$, both $(A \setminus \{a_1, \dots, a_i\}) \cup \{b_1, \dots, b_i\}$ and $(B \setminus \{b_1, \dots, b_i\}) \cup \{a_1, \dots, a_i\}$ are bases.

When $X = \{a\}$ and $Y = \{b\}$ we call the pair (a, b) a *symmetric exchange relative to A and B* and say that a and b are *symmetrically exchangeable*.

Note that Conjecture 1.2 states that any pair of bases (A, B) of a matroid M is a serial symmetric exchange (relative to themselves).

In terms of [6] a serial symmetric exchange is a sequence of one-element sets $\{a_i\}$ and $\{b_i\}$, $i = 1, \dots, k$, that constitutes a serial A -exchange and a serial B -exchange simultaneously.

We list two well-known properties of symmetric exchanges:

Observation 2.2. *$a \in A$ and $b \in B$ are symmetrically exchangeable relative to A and B if and only if $a \in C(A, b)$ and $b \in C(B, a)$.*

Observation 2.3. *Given two bases A and B of a matroid M , for each $a \in A$ there exists $b \in B$, so that a is symmetrically exchangeable with b relative to A and B .*

Definition 2.4. We call two symmetric exchanges (a_1, b_1) and (a_2, b_2) (both relative to the same two bases) *disjoint* if $a_1 \neq a_2$ and $b_1 \neq b_2$.

Assume that all elements of A are symmetrically exchangeable with the same element $b \in B$. Let $b' \neq b$ be another element of B . By Observation 2.3 there exists $a' \in A$ so that (b', a') is a symmetric exchange relative to B and A . Let $a \in A$ be such that $a \neq a'$. Then (a, b) and (a', b') are two disjoint symmetric exchanges. We conclude:

Proposition 2.5. *If A and B are two disjoint bases of a matroid M with $\rho(M) > 1$, then there always exist at least two disjoint symmetric exchanges relative to A and B .*

Now suppose $\rho(M) = 3$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. By Proposition 2.5 we may assume that (a_1, b_1) and (a_3, b_3) are disjoint symmetric exchanges relative to A and B . Exchanging the pair (a_1, b_1) relative to A and B we obtain the two bases $A' = \{b_1, a_2, a_3\}$ and $B' = \{a_1, b_2, b_3\}$. Exchanging the pair (a_3, b_3) relative to A and B we obtain the two bases $A'' = \{b_1, b_2, a_3\}$ and $B'' = \{a_1, a_2, b_3\}$. A'' and B'' can also be obtained by performing the exchange (a_2, b_2) relative to A' and B' . Thus, as is already known (see [6]), Conjecture 1.2 holds in the case where $\rho(M) = 3$.

The following lemma, also known as the circuit elimination axiom, (see [13] or [10]) will be used frequently here for proving exchange properties of bases:

Lemma 2.6. *If C_1 and C_2 are two circuits so that $x \in C_1 \cap C_2$ and $y \in C_1 \setminus C_2$, then there exists a circuit $C_3 \subset C_1 \cup C_2$ such that $x \notin C_3$ and $y \in C_3$.*

We look at the directed bipartite graph whose parts are A and B and there is an edge from $a \in A$ to $b \in B$ if and only if $b \in C(B, a)$ and an edge from $b \in B$ to $a \in A$ if and only if $a \in C(A, b)$. For $a', a'' \in A$, we look at the directed paths of length two from a' to a'' and consider the middle elements of these paths (the elements of B that "connect" a' to a''), for which we introduce the following notation:

Notation 2.7. Let A and B be two disjoint bases of a matroid M and let $a', a'' \in A$ be two distinct elements. Let

$$\text{Conn}(a', a'', A, B) = \{b \in B \mid b \in C(B, a') \text{ and } a'' \in C(A, b)\}$$

Proposition 2.8. *For any $a', a'' \in A$, $|\text{Conn}(a', a'', A, B)| \neq 1$*

Proof. By restricting M to $A \cup B$ we may assume that M is a block matroid. Thus $\text{Conn}(a', a'', A, B)$ is the intersection of the circuit $C^+(B, a')$ and the cocircuit $\{b \in B \mid a'' \in C(b, A)\} \cup \{a''\}$. The result follows from the fact that the intersection of a circuit and a cocircuit is never a singleton. \square

When a serial exchange relative to the base B is carried out and some a_i s replace b_i s in B , one by one, it is natural to ask how the "serial" supports $C(B - b_1 + a_1 - \dots - b_{i-1} + a_{i-1}, a_i)$ are related to the "original" supports $C(B, a_i)$. The following lemma, which may have its own interest, describes such a relation:

Lemma 2.9. *Let $(X = \{a_1 \prec \dots \prec a_m\}, Y = \{b_1 \prec \dots \prec b_m\})$ be a serial exchange relative to B . Let $A_0 = B_0 = \emptyset$ and for $k = 1, \dots, m$ let $A_k = \{a_1 \prec \dots \prec a_k\}$ and*

$B_k = \{b_1 \prec \dots \prec b_k\}$. Then

$$(2.1) \quad \bigcup_{i=1}^k C(B, a_i) = \bigcup_{i=1}^k C((B \setminus B_{i-1}) \cup A_{i-1}, a_i) \cap B \quad (k = 1, \dots, m).$$

Proof. We prove, by induction on k , that the set on left of (2.1) is contained in the set on the right. Let $b \in \bigcup_{i=1}^k C(B, a_i)$. If $b \in \bigcup_{i=1}^{k-1} C(B, a_i)$, then $b \in \bigcup_{i=1}^{k-1} C((B \setminus B_{i-1}) \cup A_{i-1}, a_i)$ by the induction hypothesis. Hence we may assume that $b \notin C(B, a_i)$ for all $i < k$ and $b \in C(B, a_k)$. If $b \in C((B \setminus B_{k-1}) \cup A_{k-1}, a_k)$, then we are done, so we assume $b \in C(B, a_k) \setminus C((B \setminus B_{k-1}) \cup A_{k-1}, a_k)$. We now construct a recursively defined sequence of circuits D_i , $i = k-1, k-2, \dots$. By Lemma 2.6, there is a circuit D_{k-1} in $C(B, a_k) \cup C((B \setminus B_{k-1}) \cup A_{k-1}, a_k)$ such that $b \in D_{k-1}$, but $a_k \notin D_{k-1}$. Hence $D_{k-1} \subseteq A_{k-1} \cup B$. If $b \in C((B \setminus B_{k-2}) \cup A_{k-2}, a_{k-1})$, then we are done. So we assume that $b \in D_{k-1} \setminus C((B \setminus B_{k-2}) \cup A_{k-2}, a_{k-1})$. If $a_{k-1} \notin D_{k-1}$ let $D_{k-2} = D_{k-1}$. Otherwise, we apply Lemma 2.6 on the circuits D_{k-1} and $C^+((B \setminus B_{k-2}) \cup A_{k-2}, a_{k-1})$ to obtain a circuit D_{k-2} containing b and excluding a_{k-1} . We proceed in the same manner, obtaining a sequence of circuits $D_{k-i} \subset A_{k-i}$ for $i = 1, 2, \dots$. The process must terminate by finding some $j < k$ such that $b \in C((B \setminus B_{j-1}) \cup A_{j-1}, a_j)$. Otherwise we reach a contradiction by obtaining a circuit D_j containing only one element a_i with $i < k$, contradicting the assumption that $b \notin C(B, a_i)$ for all $i < k$. The opposite containment is proved in a similar manner and is left for the reader. \square

The following lemma relates spanning sets before and after performing an exchange. It is an extension of Lemma 2 in [6]:

Lemma 2.10. *Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Suppose $B' = B - b_1 + a_1$ is a base and either $b_2 \notin C(B, a_1)$ or $b_1 \notin C(B, a_2)$. Then $b_2 \in C(B', a_2)$ if and only if $b_2 \in C(B, a_2)$.*

Proof. We show that if $b_2 \in C(B, a_2)$ and $b_2 \notin C(B', a_2)$, then $b_1 \in C(B, a_2)$ and $b_2 \in C(B, a_1)$. Clearly, $b_1 \in C(B, a_2)$, otherwise $C(B', a_2) = C(B, a_2)$, contrary to our assumption. Since $b_2 \in C(B, a_2)$, b_2 is contained in the left hand side of (2.1) (with $k = 2$). When $k = 2$ the right hand side of (2.1) consists of two terms: $C(B, a_1)$ and $C(B - b_1 + a_1, a_2) \cap B$. Since we assumed that b_2 is not contained in the second term, it must be in the first term, namely $b_2 \in C(B, a_1)$. The other direction, where we assume that $b_2 \notin C(B, a_2)$ and $b_2 \in C(B', a_2)$, is handled similarly and is left to the reader. \square

The last lemma in this section states that after performing a symmetric exchange the inserted element inherits its support and the set of elements it supports from the original one.

Lemma 2.11. *Suppose that $a \in A$ and $b \in B$ are symmetrically exchangeable relative to A and B , and let $b' \in B - b$. Then*

- (i) $C(B - b + a, b) = C(B, a) - b + a$; and
- (ii) $b \in C(A - a + b, b')$ if and only if $a \in C(A, b')$

Proof. (i) Note that $C(B, a) + a = C(B - b + a, b) + b$. Subtracting b we get (i).

(ii) Suppose that $a \notin C(A, b')$. Then the support of b' in A remains unchanged after replacing a with b . Thus $b \notin C(A - a + b, b')$. To show the other direction,

suppose $b \notin C(A - a + b, b')$. We go back and replace b with a . Again, the support of b' remains unchanged so $a \notin C(A, b')$. \square

3. SERIAL SYMMETRIC EXCHANGES

A well-known and basic result on symmetric exchanges between subsets of two bases is the following lemma ([2], [7], [14]):

Lemma 3.1. *Let A and B be two bases of a matroid M . For any $A_1 \subset A$ there exists $B_1 \subset B$ such that $(A \setminus A_1) \cup B_1$ and $(B \setminus B_1) \cup A_1$ are bases.*

We conjecture that the subsets A_1 and B_1 in Lemma 3.1 can be exchanged serially:

Conjecture 3.2. *Let A and B be two bases of a matroid M . For any $A_1 \subset A$ there exists $B_1 \subset B$ such that A_1 and B_1 form a serial symmetric exchange relative to A and B .*

the main result of this paper shows that Conjecture 3.2 holds for subsets of size two:

Theorem 3.3. *Let A and B be two bases of a matroid M . For any $A_1 \subset A$ of size two there exists $B_1 \subset B$ such that A_1 and B_1 form a serial symmetric exchange relative to A and B .*

Proof. Let $\{a_1, a_2\} \subset A$ and suppose a_1 is symmetrically exchangeable, relative to A and B , with $b_1 \in B$. Let $A' = A - a_1 + b_1$ and $B' = B - b_1 + a_1$. Assume that a_2 is not symmetrically exchangeable, relative to A' and B' , with any of $\{b_2, b_3, \dots\}$ (otherwise we are done). Hence, by Observation 2.3, a_2 must be symmetrically exchangeable with a_1 (relative to A' and B'). Thus $a_2 \in C(A', a_1)$ and $a_1 \in C(B', a_2)$. Since $a_1 \in C(B', b_1)$ and $b_1 \in C(A', a_1)$ (b_1 and a_1 are symmetrically exchangeable relative to A' and B'), $a_1 \in \text{Conn}(b_1, a_2, A', B')$ and $a_1 \in \text{Conn}(a_2, b_1, A', B')$ (Notation 2.7). By Proposition 2.8 there must be some b_i with $i \neq 1$, say b_2 , such that

$$(3.1) \quad b_2 \in C(B', b_1) \quad \text{and} \quad a_2 \in C(A', b_2),$$

and there is some b_j with $j \geq 3$, say b_3 , such that

$$(3.2) \quad b_3 \in C(B', a_2) \quad \text{and} \quad b_1 \in C(A', b_3)$$

($j \neq 2$ since we assume that a_2 is not symmetrically exchangeable with b_2 relative to A' and B').

We will show that $\{a_1, a_2\}$ and $\{b_2, b_3\}$ form a serial symmetric exchange as desired.

Since we assumed that a_2 is not symmetrically exchangeable with either b_2 or b_3 relative to A' and B' , it follows from (3.1), (3.2) and Observation 2.2 that

$$(3.3) \quad b_2 \notin C(B', a_2) \quad \text{and} \quad a_2 \notin C(A', b_3),$$

The remainder of the proof is as follows. We distinguish two separate cases. If b_1 and b_2 are symmetrically exchangeable relative to A' and B' we exchange them and show that after this exchange a_2 becomes symmetrically exchangeable with b_3 and we are done. If b_1 and b_2 are not symmetrically exchangeable relative to A' and B' , we exchange a_2 and a_1 , relative to A' and B' , and show that after this exchange

b_1 becomes symmetrically exchangeable with b_2 . We exchange them and show that we can proceed from here as in case 1.

Case 1: b_1 and b_2 are symmetrically exchangeable relative to A' and B' . Thus, by Observation 2.2,

$$(3.4) \quad b_1 \in C(A', b_2).$$

We look at the circuits $C^+(A', b_2)$ and $C^+(A', b_3)$. From (3.1), (3.2), (3.3), (3.4) and Lemma 2.6 it follows that there is a circuit D containing a_2 and excluding b_1 . D must contain at least one of b_2 and b_3 , otherwise it would contain only elements of A . We show that it must contain both. If $b_2 \notin D$, then D consists of b_3 , a_2 and possibly some other a_i s with $i > 2$. It follows that $D - b_3 = C(A', b_3)$. In particular $a_2 \in C(A', b_3)$, contrary to (3.3). If $b_3 \notin D$, then D consists of b_2 , a_2 and possibly some other a_i s with $i > 2$. It follows that $D - b_2 = C(A', b_2)$. In particular $b_1 \notin C(A', b_2)$, contrary to (3.4).

We now perform the exchange between b_1 and b_2 , relative to A' and B' , and obtain the bases $A^* = A' - b_1 + b_2$ and $B^* = B' - b_2 + b_1$. Note that A^* and B^* can be obtained from A and B by the single exchange (a_1, b_2) . Now, since the circuit D from the previous paragraph consists of b_2 , b_3 , a_2 and possibly some other a_i s with $i > 2$, we have that $D - b_3 = C(A^*, b_3)$. In particular

$$(3.5) \quad a_2 \in C(A^*, b_3).$$

Since $b_2 \notin C(B', a_2)$ (by (3.3)) we have that $C(B', a_2) = C(B^*, a_2)$ and from (3.2) we obtain

$$(3.6) \quad b_3 \in C(B^*, a_2).$$

Equations (3.5) and (3.6) imply that a_2 and b_3 are symmetrically exchangeable relative to A^* and B^* . Since A^* and B^* can be obtained from A and B by the single symmetric exchange (a_1, b_2) , we conclude that the sets $\{a_1, a_2\}$ and $\{b_2, b_3\}$ constitute a serial symmetric exchange relative to A and B .

Case 2: b_1 and b_2 are not symmetrically exchangeable relative to A' and B' . From (3.1) and Observation 2.2 we must have that

$$(3.7) \quad b_1 \notin C(A', b_2).$$

We recall that $C^+(A', a_1)$ consists, besides a_1 , of b_1 , a_2 and possibly other a_i s with $i > 2$. Also, $C^+(A', b_2)$ consists of a_2 and possibly other a_i s with $i > 2$, but excludes b_1 , by (3.7). Applying Lemma 2.6 to these two circuits we obtain a circuit D' containing a_1 , possibly other a_i s with $i > 2$, at least one of b_1 and b_2 , and excluding a_2 . We claim that D' must contain both b_1 and b_2 . First we observe that since $C(A', b_2)$ contains a_2 and excludes b_1 , $a_2 \in C(A', b_2) = C(A, b_2)$. Suppose $b_1 \notin D'$. Then D' consists of b_2 , a_1 and possibly some other a_i s with $i > 2$. Hence $D' - b_2 = C(A, b_2)$, which means that $a_2 \notin C(A, b_2) = C(A', b_2)$, contradicting (3.1). Now suppose $b_2 \notin D'$. Then D' consists of b_1 , a_1 and possibly some other a_i s with $i > 2$. Thus $D' - a_1 = C(A', a_1)$ and hence $a_2 \notin C(A', a_1)$, contradicting the assumption that a_2 and a_1 are symmetrically exchangeable relative to A' and B' . Hence $b_1, b_2 \in D'$.

We now exchange a_2 and a_1 relative to A' and B' and obtain the bases $A'' = A' - a_2 + a_1$ and $B'' = B' - a_1 + a_2$. Note that A'' and B'' can be obtained from A and B by the single exchange (a_2, b_1) . Since $a_1, b_1, b_2 \in D'$ and $a_2 \notin D'$, then

$D' - b_2 = C(A'', b_2)$. Thus

$$(3.8) \quad b_1 \in C(A'', b_2).$$

Recall that in (3.1) we had that $b_2 \in C(B', b_1)$ and in (3.3) we had that $b_2 \notin C(B', a_2)$. It follows from Lemma 2.10 that after exchanging a_2 and a_1 relative to A' and B' we must have that

$$(3.9) \quad b_2 \in C(B'', b_1).$$

From (3.8), (3.9) we have that b_1 and b_2 are symmetrically exchangeable relative to A'' and B'' .

In (3.2) we had that $b_1 \in C(A', b_3)$ and in (3.3) we had that $a_2 \notin C(A', b_3)$. It follows from Lemma 2.10 that after exchanging a_2 with a_1 , relative to A' and B' , we must have that $b_1 \in C(A'', b_3)$. By Lemma 2.11, $C(B'', a_1) = C(B', a_2) - a_1 + a_2$ and for $b' \in B' - a_1$, $a_1 \in C(A'', b')$ if and only if $a_2 \in C(A', b')$. Thus $b_3 \in C(B'', a_1)$ and $a_1 \in C(A'', b_2)$ and we are back in the setup of Case 1 with A' and B' replaced by A'' and B'' respectively, and exchanging the roles of a_1 and a_2 . Following the arguments of Case 1 we obtain the bases $A^{**} = A'' - b_1 + b_2$ and $B^{**} = B'' - b_2 + b_1$ and the symmetric exchange (a_1, b_3) relative to A^{**} and B^{**} . Since A^{**} and B^{**} can be obtained from A and B by the single symmetric exchange (a_2, b_2) , we conclude that the sets $\{a_1, a_2\}$ and $\{b_2, b_3\}$ constitute a serial symmetric exchange relative to A and B . This completes the proof of Theorem 3.3. \square

4. THE CASE $\rho(M) = 4$

In this section M is a matroid of rank 4 consisting of two bases $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

Proposition 4.1. *Let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ be a serial symmetric exchange relative to A and B . Let $a' \in A \setminus \{a_1, a_2\}$ and $b' \in B \setminus \{b_1, b_2\}$. If (a', b') is a symmetric exchange relative to A and B , then the pair (A, B) is a serial symmetric exchange.*

Proof. Let $A' = \{b_1, b_2, a_3, a_4\}$ and $B' = \{a_1, a_2, b_3, b_4\}$ be the bases obtained after serially exchanging $\{a_1, a_2\}$ with $\{b_1, b_2\}$. Suppose that (a_4, b_4) is a symmetric exchange relative to A and B , so that $\{a_1, a_2, a_3, b_4\}$ and $\{b_1, b_2, b_3, a_4\}$ are bases. These two bases can also be obtained by performing the exchange (a_3, b_3) on the bases A' and B' . \square

Theorem 4.2. *Conjecture 1.2 holds when $\rho(M) = 4$.*

Proof. From Theorem 3.3 we may assume, without loss of generality, that the sets $\{a_1, a_2\}$ and $\{b_1, b_2\}$ constitute a serial symmetric exchange relative to A and B . If among the remaining elements there is a symmetric exchange relative to A and B we are done, by Proposition 4.1. So, we assume that there is no symmetric exchange relative to A and B among $\{a_3, a_4\}$ and $\{b_3, b_4\}$. By Theorem 3.3, the pair $\{a_3, a_4\}$ and some pair of B -elements form a serial symmetric exchange. This pair must exclude at least one of b_3 and b_4 (since there is no symmetric exchange relative to A and B between $\{a_3, a_4\}$ and $\{b_3, b_4\}$, the first exchange must involve either b_1 or b_2). After serially exchanging $\{a_3, a_4\}$ with a pair of elements of B we are left with $\{a_1, a_2\}$ on the A side, and at least one of b_3 and b_4 on the B side. These remaining elements must contain a symmetric exchange relative to A and B , since both b_3 and b_4 have symmetric exchanges with a_1 or a_2 relative to B and

A (they have no symmetric exchange with either a_3 or a_4 relative to B and A). Hence, by Proposition 4.1, A and B form a serial symmetric exchange. \square

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